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On the spin–spin correlation functions of the XXZ spin- $\frac{1}{2}$ infinite chain

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Abstract

We obtain a new multiple integral representation for the spin–spin correlation functions of the XXZ spin- $\frac{1}{2}$ infinite chain. We show that this representation is closely related with the partition function of the six-vertex model with domain wall boundary conditions.

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1. Introduction

The calculation of the correlation functions of the spin chains and, in particular, their asymptotic analysis are very important problems in the field of quantum integrable models. In this paper, we consider one of the most representative examples of the lattice integrable models: the spin- $\frac{1}{2}$ Heisenberg chain. This model describes spin- $\frac{1}{2}$ particles situated in the sites of a one-dimensional lattice, interacting with their nearest neighbours,

$$H = \sum_{m=1}^M (\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta (\sigma_m^z \sigma_{m+1}^z - 1)). \quad (1.1)$$

Here, Δ is the anisotropy parameter and $\sigma_m^{x,y,z}$ are Pauli matrices associated with each site of the chain. This Hamiltonian acts in a tensor product of M two-dimensional local quantum spaces \mathcal{H}_m .⁶ We imposed here the periodic boundary conditions

$$\sigma_{M+1}^a = \sigma_1^a.$$

⁵ On leave of absence from the Steklov Institute at St Petersburg, Russia.

⁶ All the results of the present paper (with slight modifications) can also be applied to the XXZ model with external magnetic field. However, for simplicity reasons, we consider here only the case of zero magnetic field.

For simplicity reasons, we consider here only the case when the number of sites M is even.

Our main goal is to obtain some explicit expressions for the two-point spin–spin correlation functions for the infinite chain at zero temperature. They can be defined as ground-state mean values of products of local operators:

$$g_{zz}(m) = \langle \psi_g | \sigma_{m+1}^z \sigma_1^z | \psi_g \rangle, \quad (1.2)$$

$$g_{+-}(m) = \langle \psi_g | \sigma_{m+1}^+ \sigma_1^- | \psi_g \rangle, \quad (1.3)$$

where $|\psi_g\rangle$ is the ground state for the Hamiltonian (1.1).

To solve this model different techniques can be used. Originally, the energy levels were calculated in 1958 by means of the coordinate Bethe ansatz [1–4]. Later (in 1979), an algebraic version of the Bethe ansatz (or *quantum inverse scattering method*) was introduced by Faddeev, Sklyanin and Takhtajan [5]. Both methods permit one to calculate the energy level and to obtain representations for the eigenstates and, in particular, for the ground state. However, for many years, computation of the correlation functions has been possible only for the free-fermion point $\Delta = 0$ [6–10].

First explicit results for more general situations, namely for the massive antiferromagnetic regime ($\Delta > 1$), were obtained in 1992 by means of a completely different approach (q -vertex operator method) by the Kyoto group [11, 12]. In 1996, a similar conjecture was written for the critical regime of the XXZ chain ($-1 < \Delta \leq 1$) [13]. The correlation functions were represented as multiple integrals with number of integrals equal to the distance. These results were confirmed (and generalized for the XXZ chain in a constant external magnetic field) in 1999 [14, 15] by means of the algebraic Bethe ansatz and resolution of the *quantum inverse problem* [14, 16]. This new approach permitted one to better understand the results for the correlation functions and to obtain an asymptotic formula for a very particular correlation function (the so-called *emptiness formation probability*, which is the probability of finding a ferromagnetic string of length m in the ground state) [17, 18], and even an explicit result for this quantity for the point $\Delta = \frac{1}{2}$ [19]. Recently, several important new results for the correlation functions of the spin chains were obtained in a different way, from the q -KZ equations. It was shown that the multiple integrals can be separated and computed for the XXX chain [20]. Later, similar results were obtained for the XXZ [21] and XYZ [22] cases. This approach permits one to calculate the correlation functions for short distances [23] but it seems to be rather difficult to apply this method for the study of the long distance behaviour.

It is necessary to underline that the correlation functions of the XXZ chain calculated in [11, 15, 21] are not the two-point functions defined by (1.2) and (1.3) but they are the so-called ‘elementary blocks’ (i.e., the mean values of products of local elementary 2×2 matrices with only one non-zero entry in consecutive sites). The two-point functions can be expressed as sums of these elementary blocks but the number of terms in this sum grows exponentially with respect to the distance (2^{m+1}), which is very inconvenient for the computation of the long distance asymptotics. The main goal of the present paper is to reduce our expression for the two-point functions to a sum of $m + 1$ terms and thus to obtain a compact and manageable representation for it. One way to solve this problem was proposed in [24], and here we present an alternative way to simplify the expressions for the spin–spin correlation functions. The advantage of this alternative approach is a simpler final form of the terms (each of them contains the same number of integrals and there is no need to introduce any additional integrations). It is also important to mention that each term in the final sum has a form very similar to the symmetric representation for the emptiness formation probability, obtained in [24], and thus

there is a possibility to use the saddle point method similarly to [17] for its rough asymptotic analysis.

The paper is organised as follows. In the next section, we give a short reminder of the algebraic Bethe ansatz solution of the XXZ chain (following [5]) and an overview of our way to calculate the elementary blocks following our paper [15]. In section 3, we introduce the generating function of the correlation functions of the third components of spin and we show how to express this quantity in terms of the elementary blocks. We also explain how this function is related with the two-point function (1.2). In section 4, we prove our main result for the generating function and show how a similar technique can be applied directly to the correlation functions (1.3) and (1.2). A further re-summation is performed in section 5, which permits us to reduce the result to only one term. In the last section, we show how this method can be applied in the free-fermion point to obtain the known results for the two-point functions and their asymptotics.

2. XXZ chain: algebraic Bethe ansatz and elementary blocks

First, we have to determine the ground state of the Hamiltonian (1.1). In the framework of the algebraic Bethe ansatz [5], it can be described in terms of the generalized creation and annihilation operators which can be obtained as elements of the quantum monodromy matrix. This matrix is completely defined by the R -matrix of the model which, for the XXZ chain, is the usual trigonometric solution of the Yang–Baxter equation, acting in the tensor product of two auxiliary spaces $V_1 \otimes V_2$, $V_i = \mathbb{C}^2$:

$$R(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sinh \lambda}{\sinh(\lambda-i\zeta)} & -\frac{i \sin \zeta}{\sinh(\lambda-i\zeta)} & 0 \\ 0 & -\frac{i \sin \zeta}{\sinh(\lambda-i\zeta)} & \frac{\sinh \lambda}{\sinh(\lambda-i\zeta)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.1}$$

the parameter ζ being related to the anisotropy parameter⁷

$$\Delta = \cos \zeta.$$

The monodromy matrix can be constructed as a product of R -matrices and can be written as a 2×2 matrix in the auxiliary space:

$$T(\lambda) = R_{0M} \left(\lambda - \xi_N + i \frac{\zeta}{2} \right) \cdots R_{01} \left(\lambda - \xi_1 + i \frac{\zeta}{2} \right) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_{[0]}. \tag{2.2}$$

Here, $\{\xi\}$ is a set of arbitrary inhomogeneity parameters and the R -matrices R_{0k} act in the tensor product of the auxiliary space V_0 and the local quantum space \mathcal{H}_k . The operator entries of the monodromy matrix A , B , C and D act in the same quantum space \mathcal{H} as the Hamiltonian of the XXZ chain. The commutation relations of these operators can be obtained from the Yang–Baxter equation:

$$R_{12}(\lambda - \mu) T_1(\lambda) T_2(\mu) = T_2(\mu) T_1(\lambda) R_{12}(\lambda - \mu). \tag{2.3}$$

From this relation, one can also easily see that traces of the monodromy matrix taken in the auxiliary space (*transfer matrices*) commute for any values of the spectral parameters,

$$[A(\lambda) + D(\lambda), A(\mu) + D(\mu)] = 0. \tag{2.4}$$

⁷ We give all the formulae in this paper for the critical regime ($-1 < \Delta < 1$) of the XXZ chain, but it will be clear that similar computations can be done for the massive regime ($\Delta > 1$) and for the XXX chain ($\Delta = 1$).

The Hamiltonian of the XXZ model can be reconstructed from the transfer matrix in the homogeneous limit when all the inhomogeneity parameters are equal: $\xi_k = 0$,

$$H = c \left. \frac{\partial}{\partial \lambda} \log(A(\lambda) + D(\lambda)) \right|_{\lambda=0} + \text{const.} \quad (2.5)$$

It means in particular that the Hamiltonian commutes with the transfer matrix for any value of the spectral parameter,

$$[H, A(\lambda) + D(\lambda)] = 0,$$

and the eigenstates of the transfer matrix (for arbitrary λ) are eigenstates of the Hamiltonian.

To construct the eigenstates of the transfer matrix one can use the operators $B(\lambda)$ as creation operator (and operators $C(\lambda)$ as annihilation operators). This is possible if there exists a reference state $|0\rangle$ which is an eigenstate of the operators $A(\lambda)$ and $D(\lambda)$ and is annihilated by the operators $C(\lambda)$ for any value of λ . For the XXZ chain such a state exists and it is the ferromagnetic state with all the spins up. Now other eigenstates can be constructed by the action of operators B on this ferromagnetic state. More precisely, using the Yang–Baxter algebra (2.3), one can show that the eigenstates of the transfer matrix (and hence of the Hamiltonian in the homogeneous case) can be constructed in the form

$$|\psi\rangle = B(\lambda_1) \cdots B(\lambda_N)|0\rangle, \quad (2.6)$$

where the spectral parameters $\{\lambda_j\}$ satisfy the *Bethe equations*,

$$\prod_{m=1}^M \frac{\sinh(\lambda_j - \xi_m + i\frac{\zeta}{2})}{\sinh(\lambda_j - \xi_m - i\frac{\zeta}{2})} \cdot \prod_{\substack{k=1 \\ k \neq j}}^N \frac{\sinh(\lambda_j - \lambda_k - i\zeta)}{\sinh(\lambda_j - \lambda_k + i\zeta)} = 1, \quad j = 1, \dots, N. \quad (2.7)$$

The Bethe equations are very difficult to solve for a finite chain. However, it can be shown that the ground state of the XXZ chain is described by this procedure in the homogeneous case with $N = \frac{M}{2}$ and can be specified by a special choice of integers in the logarithmic form of the Bethe equations:

$$p_0(\lambda_j) - \frac{1}{M} \sum_{k=1}^{M/2} \phi(\lambda_j - \lambda_k) = -\frac{\pi}{2} - \frac{\pi}{M} + \frac{2\pi j}{M}, \quad j = 1, \dots, \frac{M}{2}, \quad (2.8)$$

where the bare momentum $p_0(\lambda)$ and the ‘scattering phase’ $\phi(\lambda)$ are defined as

$$p_0(\lambda) = i \log \frac{\sinh(\lambda + i\frac{\zeta}{2})}{\sinh(\lambda - i\frac{\zeta}{2})} \quad \phi(\lambda) = i \log \frac{\sinh(\lambda + i\zeta)}{\sinh(\lambda - i\zeta)}.$$

This state is the ground state of the XXZ chain in the homogeneous case, but even for the inhomogeneous model one can define a Bethe state with this choice of integers in the right-hand side of the logarithmic Bethe equations. In the calculation of the correlation functions, it will be convenient to consider such a state first and to take the homogeneous limit only in the final result.

Even for the ground state there is no way in a generic situation to solve the Bethe equations explicitly. However, in the thermodynamic limit $M \rightarrow \infty$, this state can be described in a very simple way in terms of the *density of rapidities*. More precisely, in the thermodynamic limit, for any smooth bounded function $f(\lambda)$, any sum over the Bethe roots corresponding to the ground state can be replaced by the following integral:

$$\frac{1}{M} \sum_{j=1}^{M/2} f(\lambda_j) = \int_{-\infty}^{\infty} d\lambda \rho(\lambda) f(\lambda) + o\left(\frac{1}{M}\right), \quad (2.9)$$

where the density function $\rho(\lambda)$ can be obtained from a simple integral equation which replaces in the thermodynamic limit the Bethe equations,

$$\rho(\lambda) + \int_{-\infty}^{\infty} d\mu \rho(\mu) K(\lambda - \mu) = \frac{1}{2\pi} \rho'_0(\lambda), \tag{2.10}$$

where the kernel $K(\lambda)$ is a derivative of the ‘scattering phase’

$$K(\lambda) = \frac{1}{2\pi} \phi'(\lambda) = \frac{1}{2\pi} \frac{\sin(2\zeta)}{\sinh(\lambda + i\zeta) \sinh(\lambda - i\zeta)}. \tag{2.11}$$

This equation can be easily solved by Fourier transform,

$$\rho(\lambda) = \frac{1}{2\zeta \cosh(\frac{\pi}{\zeta}\lambda)}. \tag{2.12}$$

This information about the ground state is sufficient for the calculation of the correlation functions in the thermodynamic limit.

The first problem which arises when one tries to calculate ground-state mean values of products of local operators in the framework of the algebraic Bethe ansatz is the fact that creation (annihilation) operators $B(C)$ are non-local and do not permit a simple expansion in terms of spin operators. Thus, it is rather difficult to establish commutation relations between these two types of objects. In our paper [14], we proposed a way to solve this problem by expressing the local operators in terms of the monodromy matrix elements (hence solving the quantum inverse problem). The expression for the local elementary 2×2 matrices $E_m^{\epsilon'_m \epsilon_m}$ with only one non-zero entry,

$$E_{jk}^{\epsilon'_j \epsilon_j} = \delta_{j\epsilon'} \delta_{k\epsilon},$$

can be written in a very simple form

$$E_m^{\epsilon'_m, \epsilon_m} = \prod_{k=1}^{m-1} (A + D) \left(\xi_k - i\frac{\zeta}{2} \right) T_{\epsilon_m, \epsilon'_m} \left(\xi_m - i\frac{\zeta}{2} \right) \prod_{k=1}^m (A + D)^{-1} \left(\xi_k - i\frac{\zeta}{2} \right). \tag{2.13}$$

It is easy to see that every local operator is expressed as a monodromy matrix element dressed with transfer matrices. In particular, the operator σ_m^+ is expressed as the dressed operator $C(\xi_m - i\frac{\zeta}{2})$, σ_m^- as $B(\xi_m - i\frac{\zeta}{2})$ and σ_m^z as $A(\xi_m - i\frac{\zeta}{2}) - D(\xi_m - i\frac{\zeta}{2})$. It is important to mention that this solution is very convenient to calculate the ground-state mean values as the ground state is an eigenstate of the transfer matrix,

$$\left(A \left(\xi_k - i\frac{\zeta}{2} \right) + D \left(\xi_k - i\frac{\zeta}{2} \right) \right) |\psi_g\rangle = \prod_{j=1}^{M/2} \frac{\sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})}{\sinh(\lambda_j - \xi_k + i\frac{\zeta}{2})} |\psi_g\rangle.$$

Now the correlation functions can be expressed only in terms of the monodromy matrix elements. The first type of object we can consider using this approach are the ‘elementary blocks’, i.e., the ground-state mean values of products of the elementary local matrices in m consecutive sites,

$$F_m(\{\epsilon_j, \epsilon'_j\}) = \langle \psi_g | \prod_{j=1}^m E_j^{\epsilon'_j, \epsilon_j} | \psi_g \rangle. \tag{2.14}$$

Using the solution of the quantum inverse problem such quantities can be expressed only in terms of the monodromy matrix elements,

$$F_m(\{\epsilon_j, \epsilon'_j\}) = \left(\prod_{k=1}^m \prod_{j=1}^{M/2} \frac{\sinh(\lambda_j - \xi_k + i\frac{\zeta}{2})}{\sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})} \right) \times \frac{\langle 0 | \prod_{j=1}^{M/2} C(\lambda_j) \prod_{k=1}^m T_{\epsilon_k, \epsilon'_k}(\xi_k - i\frac{\zeta}{2}) \prod_{j=1}^{M/2} B(\lambda_j) | 0 \rangle}{\langle 0 | \prod_{j=1}^{M/2} C(\lambda_j) \prod_{j=1}^{M/2} B(\lambda_j) | 0 \rangle}, \tag{2.15}$$

(it is important to mention that operators $B(\lambda)$ are not normalized and thus to obtain the mean value, one should divide the rhs by the norms of the Bethe vector corresponding to the ground state). Now we can use the Yang–Baxter algebra, i.e., the commutation relations between the monodromy matrix elements. In particular, we can act with the monodromy matrix elements on the dual Bethe state constructed by the actions of operators $C(\lambda)$.

It is easy to see that after acting with all the operators in (2.15) on the dual Bethe state one obtains again a sum of states constructed by the action of the operators $C(\lambda)$ on the dual ferromagnetic state (but the spectral parameters no longer satisfy the Bethe equations). Then, using the Gaudin–Korepin formula for the norm of the Bethe states [27–29], and the determinant representation for the scalar products of a Bethe state with an arbitrary state [30, 14], one can express the correlation functions (2.14) as sums of determinants. Now we have an explicit formula for the elementary blocks in term of the ground-state solution of the Bethe equations. In the thermodynamic limit, this sum can be simplified as all the sums over the Bethe roots can be replaced by integrals with density. The final result for any elementary block can be written as multiple integrals

$$F_m(\{\epsilon_j, \epsilon'_j\}) = \frac{1}{\prod_{j>k} \sinh(\xi_j - \xi_k)} \int_{-\infty}^{\infty} d\lambda_1 \cdots \int_{-\infty}^{\infty} d\lambda_m \mathcal{F}(\{\lambda_k\}, \{\xi_j, \epsilon_j, \epsilon'_j\}) \det_m S(\{\lambda_j\}, \{\xi_k\}). \quad (2.16)$$

Here, the $m \times m$ matrix S does not depend on the choice of local operators and is defined uniquely by the ground state. Its elements can be written in terms of the density function (2.12)

$$S_{jk} = \rho(\lambda_j - \xi_k). \quad (2.17)$$

The algebraic part $\mathcal{F}(\{\lambda_k\}, \{\xi_j, \epsilon_j, \epsilon'_j\})$ arises from the commutation relation of the monodromy matrix elements and does not depend on the ground state. The expression for this function for the most general case can be found in the paper [15]. Here, for the calculation of the two-point functions we need only some particular blocks and we will give an explicit expression for the corresponding algebraic parts in the next sections.

Once the elementary blocks are calculated, any correlation function can be written as a sum of such multiple integrals. However, the sums which appear for the two-point functions contain a number of terms that grows exponentially with the distance and in such a form cannot be used for the asymptotic analysis. For example, using the solution of the quantum inverse problem for the correlation functions of the third components of spin $g_{zz}(m)$, we obtain the following expression:

$$g_{zz}(m) = \left(\prod_{k=1}^{m+1} \prod_{j=1}^{M/2} \frac{\sinh(\lambda_j - \xi_k + i\frac{\zeta}{2})}{\sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})} \right) \langle \psi_g | \left(A \left(\xi_1 - i\frac{\zeta}{2} \right) - D \left(\xi_1 - i\frac{\zeta}{2} \right) \right) \times \left(\prod_{j=2}^m \left(A \left(\xi_j - i\frac{\zeta}{2} \right) + D \left(\xi_j - i\frac{\zeta}{2} \right) \right) \right) \times \left(A \left(\xi_{m+1} - i\frac{\zeta}{2} \right) - D \left(\xi_{m+1} - i\frac{\zeta}{2} \right) \right) | \psi_g \rangle. \quad (2.18)$$

It is easy to see that this function can be rewritten as a sum of 2^{m+1} elementary blocks. Similarly, for the function $g_{+-}(m)$, one has

$$g_{+-}(m) = \left(\prod_{k=1}^{m+1} \prod_{j=1}^{M/2} \frac{\sinh(\lambda_j - \xi_k + i\frac{\zeta}{2})}{\sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})} \right) \langle \psi_g | C \left(\xi_1 - i\frac{\zeta}{2} \right) \times \left(\prod_{j=2}^m \left(A \left(\xi_j - i\frac{\zeta}{2} \right) + D \left(\xi_j - i\frac{\zeta}{2} \right) \right) \right) B \left(\xi_{m+1} - i\frac{\zeta}{2} \right) | \psi_g \rangle, \quad (2.19)$$

and it can be written as a sum of 2^{m-1} elementary blocks. The main goal of this paper is to obtain a manageable expression for these correlation functions from the multiple integral representation for the elementary blocks.

3. Generating function

In this section, we consider the correlation function of the third components of spin $g_{zz}(m)$. To calculate this function it is convenient to introduce a new object: the *generating function*

$$\mathcal{Q}_m(\beta) \equiv \langle \psi_g | \exp\{\beta \mathcal{Q}_{1,m}\} | \psi_g \rangle, \quad \mathcal{Q}_{1,m} = \sum_{j=1}^m \frac{1}{2} (1 - \sigma_j^z). \quad (3.1)$$

To obtain the two-point function from the generating function one should take its second derivative on β and second lattice derivative on m

$$g_{zz}(m) = \left(2\mathcal{D}_m^2 \frac{\partial^2}{\partial \beta^2} - 4\mathcal{D}_m \frac{\partial}{\partial \beta} + 1 \right) \mathcal{Q}_m(\beta) \Big|_{\beta=0}, \quad (3.2)$$

where we used the standard definition of the first and second lattice derivatives

$$\mathcal{D}_m f(m) = f(m+1) - f(m), \quad \mathcal{D}_m^2 f(m) = f(m+1) - 2f(m) + f(m-1).$$

Using the solution of the quantum inverse problem, we can rewrite this quantity in terms of the monodromy matrix elements:

$$\mathcal{Q}_m(\beta) = \left(\prod_{k=1}^m \prod_{j=1}^{M/2} \frac{\sinh(\lambda_j - \xi_k + i\frac{\zeta}{2})}{\sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})} \right) \langle \psi_g | \left(\prod_{j=1}^m \left(A \left(\xi_j - i\frac{\zeta}{2} \right) + e^\beta D \left(\xi_j - i\frac{\zeta}{2} \right) \right) \right) | \psi_g \rangle. \quad (3.3)$$

The generating function can be expressed as a sum of 2^m elementary blocks containing only diagonal elements of the monodromy matrix (operators $A(\xi - i\frac{\zeta}{2})$ and $D(\xi - i\frac{\zeta}{2})$). For such elementary blocks, the algebraic part can be written in a factorized form, as a product of ‘two-particle contributions’,

$$\begin{aligned} & \left(\prod_{k=1}^m \prod_{j=1}^{M/2} \frac{\sinh(\lambda_j - \xi_k + i\frac{\zeta}{2})}{\sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})} \right) \langle \psi_g | T_{a_1 a_1} \left(\xi_1 - i\frac{\zeta}{2} \right) \cdots T_{a_m a_m} \left(\xi_m - i\frac{\zeta}{2} \right) | \psi_g \rangle \\ &= \frac{1}{\prod_{j < k} \sinh(\xi_j - \xi_k)} \int_{-\infty}^{\infty} d\lambda_1 \cdots \int_{-\infty}^{\infty} d\lambda_m \det_m S(\{\lambda_k\}, \{\xi_j\}) \\ & \times \prod_{j > k} \frac{\sinh(\lambda_j - \xi_k + i\epsilon_j \zeta) \sinh(\lambda_k - \xi_j - i\epsilon_k \zeta)}{\sinh(\lambda_j - \lambda_k + i(\epsilon_j + \epsilon_k) \zeta)}, \end{aligned} \quad (3.4)$$

where numbers $\epsilon_j = \pm \frac{1}{2}$ depend on the choice of the corresponding monodromy matrix elements $\epsilon_j = \frac{3}{2} - a_j$.

It is clear that the elementary blocks with fixed number of operators of each type have a rather similar structure. Thus, it is quite natural to put together the terms with the same number of operators D and to represent the generating function as power series on e^β ,

$$\mathcal{Q}_m(\beta) = \sum_{s=0}^m e^{s\beta} F_s(m). \quad (3.5)$$

The main result of this paper is a simple and compact formula for the terms $F_s(m)$. This function can be expressed in terms of elementary blocks,

$$F_s(m) = \left(\prod_{k=1}^m \prod_{j=1}^{M/2} \frac{\sinh(\lambda_j - \xi_k + i\frac{\zeta}{2})}{\sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})} \right) \times \sum_{a_1 + \dots + a_m = m-s} \langle \psi_g | T_{a_1 a_1} \left(\xi_1 - i\frac{\zeta}{2} \right) \dots T_{a_m a_m} \left(\xi_m - i\frac{\zeta}{2} \right) | \psi_g \rangle, \quad (3.6)$$

and hence the sums over all possible positions of operators D and A also can be written as multiple integrals

$$F_s(m) = \prod_{j < k} \frac{1}{\sinh(\xi_j - \xi_k)} \int_{-\infty}^{\infty} d\lambda_1 \dots \int_{-\infty}^{\infty} d\lambda_m G_s(m, \{\lambda_j\} | \{\xi_k\}) \det_m S(\{\lambda_j\}, \{\xi_k\}), \quad (3.7)$$

where the function $G_s(m, \{\lambda_j\} | \{\xi_k\})$ under the integrals is a sum over the permutations σ of the set $1, \dots, m$

$$G_s(m, \{\lambda_j\} | \{\xi_k\}) = \frac{1}{s!(m-s)!} \times \sum_{\sigma} (-1)^{[\sigma]} \prod_{j > k} \frac{\sinh(\lambda_{\sigma(j)} - \xi_k + i\epsilon_{\sigma(j)}\zeta) \sinh(\lambda_{\sigma(k)} - \xi_j - i\epsilon_{\sigma(k)}\zeta)}{\sinh(\lambda_{\sigma(j)} - \lambda_{\sigma(k)} + i(\epsilon_{\sigma(j)} + \epsilon_{\sigma(k)})\zeta)}, \quad (3.8)$$

where $(-1)^{[\sigma]}$ is the sign of the permutation σ , the factorials in the denominator arise from the additional summations over permutations of the variables of the same type (with $\epsilon_j = \frac{1}{2}$ or $\epsilon_j = -\frac{1}{2}$). We set for $j \leq s$, $\epsilon_j = -\frac{1}{2}$ and for $j > s$, $\epsilon_j = \frac{1}{2}$. We use here a very important property of the multiple integral representation, namely the fact that the determinant of densities does not depend on the choice of local operators.

The function $G_s(m, \{\lambda_j\} | \{\xi_k\})$ is a rational function of e^{λ_j} and e^{ξ_k} . It is by definition skew-symmetric under the permutations of $\lambda_1, \dots, \lambda_m$. The first property of this function which we need for our computation is its symmetry on the variables ξ_j .

Lemma 3.1. *The function $G_s(m, \{\lambda_j\} | \{\xi_k\})$ defined by (3.8) is symmetric under the permutations of the variables $\xi_1, \xi_2, \dots, \xi_m$.*

Proof. To prove this lemma it is sufficient to show for any k that

$$G_s(m, \{\lambda_j\} | \xi_1, \dots, \xi_k, \xi_{k+1}, \dots, \xi_m) - G_s(m, \{\lambda_j\} | \xi_1, \dots, \xi_{k+1}, \xi_k, \dots, \xi_m) = 0.$$

Consider first this difference for corresponding monomials in (3.8) and note that two terms differ only in one ‘two-particle contribution’:

$$\prod_{\substack{j > l \\ l \neq k}} \frac{\sinh(\lambda_j - \xi_l + i\epsilon_j\zeta) \sinh(\lambda_l - \xi_j - i\epsilon_l\zeta)}{\sinh(\lambda_j - \lambda_l + i(\epsilon_j + \epsilon_l)\zeta)} \prod_{j=k+2}^m \frac{\sinh(\lambda_j - \xi_k + i\epsilon_j\zeta) \sinh(\lambda_k - \xi_j - i\epsilon_k\zeta)}{\sinh(\lambda_j - \lambda_k + i(\epsilon_j + \epsilon_k)\zeta)} \times \left(\frac{\sinh(\lambda_{k+1} - \xi_k + i\epsilon_{k+1}\zeta) \sinh(\lambda_k - \xi_{k+1} - i\epsilon_k\zeta)}{\sinh(\lambda_{k+1} - \lambda_k + i(\epsilon_{k+1} + \epsilon_k)\zeta)} \right)$$

$$\begin{aligned}
 & - \frac{\sinh(\lambda_{k+1} - \xi_{k+1} + i\epsilon_{k+1}\zeta) \sinh(\lambda_k - \xi_k - i\epsilon_k\zeta)}{\sinh(\lambda_{k+1} - \lambda_k + i(\epsilon_{k+1} + \epsilon_k)\zeta)} \\
 = & \sinh(\xi_k - \xi_{k+1}) \prod_{\substack{j>l \\ l \neq k}} \frac{\sinh(\lambda_j - \xi_l + i\epsilon_j\zeta) \sinh(\lambda_l - \xi_j - i\epsilon_l\zeta)}{\sinh(\lambda_j - \lambda_l + i(\epsilon_j + \epsilon_l)\zeta)} \\
 \times & \prod_{j=k+2}^m \frac{\sinh(\lambda_j - \xi_k + i\epsilon_j\zeta) \sinh(\lambda_k - \xi_j - i\epsilon_k\zeta)}{\sinh(\lambda_j - \lambda_k + i(\epsilon_j + \epsilon_k)\zeta)}.
 \end{aligned}$$

Consider now the same term for the following permutation of the set $\lambda_1, \dots, \lambda_m$:

$$\{\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \lambda_k, \lambda_{k+2}, \dots, \lambda_m\}.$$

It is easy to see that for this permutation we obtain exactly the same contribution with an opposite sign (as the sign of this permutation is -1) and the sum of these two contributions is zero. Now we should take a sum of such monomials over all possible permutations of the set $\lambda_1, \dots, \lambda_m$, but as this sum can be split into such pairs with permuted $\lambda_{\sigma(k)}$ and $\lambda_{\sigma(k+1)}$ we immediately obtain that this sum is zero. \square

Using this symmetry, we can obtain recursion relations for the function $G_s(m, \{\lambda_j\}|\{\xi_k\})$ in the points $\lambda_j = \xi_k - i\frac{\zeta}{2}$. It is more convenient now to extract the common denominator and to consider the function $\tilde{G}_s(m, \{\lambda_j\}|\{\xi_k\})$:

$$\begin{aligned}
 G_s(m, \{\lambda_j\}|\{\xi_k\}) &= \frac{1}{s!(m-s)!} \prod_{j>k} \frac{\sinh(\lambda_j - \lambda_k)}{\sinh(\lambda_j - \lambda_k + i(\epsilon_j + \epsilon_k)\zeta) \sinh(\lambda_j - \lambda_k - i(\epsilon_j + \epsilon_k)\zeta)} \\
 &\times \tilde{G}_s(m, \{\lambda_j\}|\{\xi_k\}).
 \end{aligned}$$

Directly from this definition, we can establish two evident lemmas:

Lemma 3.2. *The function $e^{(m-1)\lambda_j} \tilde{G}_s(m, \{\lambda_j\}|\{\xi_k\})$ is a polynomial function of $\epsilon^{2\lambda_j}$ of degree $m - 1$.*

Lemma 3.3.

$$\tilde{G}_0(1, \lambda_1|\xi_1) = \tilde{G}_1(1, \lambda_1|\xi_1) = 1.$$

These lemmas mean that if we obtain recurrence relations for this function in the points $\lambda_j = \xi_k - i\frac{\zeta}{2}$ then they will define it completely (as it is a polynomial of degree $m - 1$ defined in m points). The recursion relation for this function can be written in the following form:

Lemma 3.4. $\tilde{G}_s(m, \{\lambda_j\}|\{\xi_k\})$ satisfies the following recursion relations:

$$\begin{aligned}
 \tilde{G}_s(m, \{\lambda_l\}|\{\xi_k\})|_{\lambda_j = \xi_k - i\frac{\zeta}{2}} &= \prod_{\substack{a=1 \\ a \neq k}}^m \sinh\left(\lambda_j - \xi_a - i\frac{\zeta}{2}\right) \prod_{a \neq j} \sinh\left(\lambda_a - \xi_k - i\frac{\zeta}{2}\right) \\
 &\times \tilde{G}_s(m-1, \lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots, \lambda_m | \xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_m), \\
 \epsilon_j &= \frac{1}{2}
 \end{aligned} \tag{3.9}$$

$$\begin{aligned}
 \tilde{G}_s(m, \{\lambda_l\}|\{\xi_k\})|_{\lambda_j = \xi_k - i\frac{\zeta}{2}} &= \prod_{\substack{a=1 \\ a \neq k}}^m \sinh\left(\lambda_j - \xi_a - i\frac{\zeta}{2}\right) \prod_{a \neq j} \sinh\left(\lambda_a - \xi_k - i\frac{\zeta}{2}\right) \\
 &\times \tilde{G}_{s-1}(m-1, \lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots, \lambda_m | \xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_m), \\
 \epsilon_j &= -\frac{1}{2}
 \end{aligned} \tag{3.10}$$

Proof. From lemma 3.1 it is clear that it is sufficient to prove these recursion relations only for $\lambda_j = \xi_1 - i\frac{\zeta}{2}$ for $\epsilon_j = \frac{1}{2}$ and for $\lambda_j = \xi_m - i\frac{\zeta}{2}$ for $\epsilon_j = -\frac{1}{2}$. It is easy to see that only terms with $\sigma(1) = j$ ($\sigma(m) = j$) will survive in the sum over permutations (3.8). We obtain a sum over permutations of the set $\{\lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots, \lambda_m\}$ and using the definition of the function $\tilde{G}_s(m, \{\lambda_l\}|\{\xi_k\})$, we obtain immediately the recursion relations (3.9), (3.10). \square

These recursion relations define completely the function $\tilde{G}_s(m, \{\lambda_l\}|\{\xi_k\})$. It means that if we can find a function satisfying lemmas 3.1–3.4 it is the function $\tilde{G}_s(m, \{\lambda_l\}|\{\xi_k\})$ we need. Such properties (without dependence on s) were established for the first time by Korepin in [29] for the partition function of the six-vertex model with domain wall boundary conditions. In fact the recursion relations obtained here are exactly the same as those of Korepin. The (unique) solution for these relations, found by Izergin in [31] (which does not depend on s), satisfies lemmas 3.1–3.4 for any value of s and thus gives the function $\tilde{G}_s(m, \{\lambda_l\}|\{\xi_k\})$:

Theorem 3.1 (Izergin 1987). *The only function satisfying lemmas 3.1–3.4 is*

$$\tilde{G}_s(m, \{\lambda_l\}|\{\xi_k\}) = \frac{1}{\sin^m \zeta} Z_m(\{\lambda_l\}|\{\xi_k\}),$$

where $Z_m(\{\lambda\}|\{\xi\})$ is the partition function of the inhomogeneous six-vertex model with domain wall boundary conditions, which can be written in the following form:

$$Z_m(\{\lambda_l\}|\{\xi_k\}) = \frac{\prod_{j=1}^m \prod_{k=1}^m \sinh(\lambda_j - \xi_k + i\frac{\zeta}{2}) \sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})}{\prod_{j>k} \sinh(\lambda_j - \lambda_k) \sinh(\xi_k - \xi_j)} \det_m \mathcal{M}(\{\lambda_l\}|\{\xi_k\}), \tag{3.11}$$

$$\mathcal{M}_{jk} = \frac{\sin \zeta}{\sinh(\lambda_j - \xi_k + i\frac{\zeta}{2}) \sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})}. \tag{3.12}$$

Proof. To prove this theorem one needs just to verify that the function $Z_m(\{\lambda\}|\{\xi\})$ satisfies all the lemmas 3.1–3.4 for any value of s . Then, as it is a polynomial of degree $m - 1$ coinciding with another polynomial of the same degree in m points, one can conclude that these two functions are equal. \square

The fact that the function $\tilde{G}_s(m, \{\lambda_l\}|\{\xi_k\})$ does not depend on s is a very important peculiarity of this re-summation technique. Hence, the generating function can be written as a sum of $m + 1$ multiple integrals of the following form:

$$F_s(m) = \frac{1}{s!(m-s)! \sin^m \zeta} \prod_{j<k} \frac{1}{\sinh(\xi_j - \xi_k)} \int_{-\infty}^{\infty} d\lambda_1 \dots \int_{-\infty}^{\infty} d\lambda_m \det_m S(\{\lambda\}, \{\xi\}) \times \Theta_m^s(\lambda_1, \dots, \lambda_m) Z_m(\{\lambda\}|\{\xi\}), \tag{3.13}$$

where the only factor depending on s which we denote by $\Theta_m^s(\{\lambda\})$

$$\Theta_m^s(\lambda_1, \dots, \lambda_m) = \prod_{k=1}^s \prod_{j=s+1}^m \frac{1}{\sinh(\lambda_j - \lambda_k)} \prod_{m \geq j > k > s} \frac{\sinh(\lambda_j - \lambda_k)}{\sinh(\lambda_j - \lambda_k + i\zeta) \sinh(\lambda_j - \lambda_k - i\zeta)} \times \prod_{s \geq j > k \geq 1} \frac{\sinh(\lambda_j - \lambda_k)}{\sinh(\lambda_j - \lambda_k + i\zeta) \sinh(\lambda_j - \lambda_k - i\zeta)} \tag{3.14}$$

will appear in all our results.

Thus, we reduced the number of terms from exponential to polynomial order. This representation is also interesting because of its unexpected relation with the partition function of the corresponding inhomogeneous six-vertex model with domain wall boundary conditions. We also suppose that this representation can be convenient for the asymptotic analysis of the two-point function $g_{zz}(m)$.

It is also very important to mention that for this re-summation we manipulated only the algebraic part of the expression for the elementary blocks and hence it can be done in a similar way for the XXZ spin chain in a magnetic field. The result in this case has a slightly more complicated determinant of densities and different integration contours but the algebraic part is the same and contains the partition function $Z_m(\{\lambda\}|\{\xi\})$.

A rough asymptotic analysis of the generating function can be performed using a modification of the saddle point technique introduced in [24] for the emptiness formation probability. It shows that there is no Gaussian contribution to this correlation function and that the main order can be written as $C \exp(\frac{m\beta}{2})$, as it should be, but it is not sufficient to describe the power-like behaviour of the two-point function. The interesting peculiarity of this saddle point analysis is the fact that the ‘saddle point density’ here coincides with the ground-state density $\rho(\lambda)$.

4. Two-point functions

A similar re-summation can be done for the two-point functions $g_{+-}(m)$ (or for $g_{xx}(m)$ and $g_{yy}(m)$) and $g_{zz}(m)$. The function $g_{+-}(m)$ can be written in the following form in terms of the monodromy matrix elements:

$$g_{+-}(m) = \left(\prod_{k=1}^{m+1} \prod_{j=1}^{M/2} \frac{\sinh(\lambda_j - \xi_k + i\frac{\zeta}{2})}{\sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})} \right) \langle \psi_g | C \left(\xi_1 - i\frac{\zeta}{2} \right) \times \left(\prod_{j=2}^m \left(A \left(\xi_j - i\frac{\zeta}{2} \right) + D \left(\xi_j - i\frac{\zeta}{2} \right) \right) \right) B \left(\xi_{m+1} - i\frac{\zeta}{2} \right) | \psi_g \rangle. \tag{4.1}$$

To calculate this function one should sum up 2^{m-1} elementary blocks of the following form:

$$g_{+-}(m) = \left(\prod_{k=1}^{m+1} \prod_{j=1}^{M/2} \frac{\sinh(\lambda_j - \xi_k + i\frac{\zeta}{2})}{\sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})} \right) \times \sum_{a_j=1,2} \langle \psi_g | C \left(\xi_1 - i\frac{\zeta}{2} \right) \left(\prod_{j=2}^m T_{a_j a_j} \left(\xi_j - i\frac{\zeta}{2} \right) \right) B \left(\xi_{m+1} - i\frac{\zeta}{2} \right) | \psi_g \rangle,$$

which can be written as multiple integrals in a factorized form

$$\begin{aligned} & \left(\prod_{k=1}^{m+1} \prod_{j=1}^{M/2} \frac{\sinh(\lambda_j - \xi_k + i\frac{\zeta}{2})}{\sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})} \right) \langle \psi_g | C \left(\xi_1 - i\frac{\zeta}{2} \right) T_{a_2 a_2} \left(\xi_2 - i\frac{\zeta}{2} \right) \\ & \quad \dots T_{a_m a_m} \left(\xi_m - i\frac{\zeta}{2} \right) B \left(\xi_{m+1} - i\frac{\zeta}{2} \right) | \psi_g \rangle \\ & = \frac{1}{\prod_{k>j\geq 1}^{m+1} \sinh(\xi_j - \xi_k)} \int_{-\infty}^{\infty} d\lambda_2 \dots \int_{-\infty}^{\infty} d\lambda_m \int_{-\infty}^{\infty} d\lambda_+ \\ & \quad \times \int_{-\infty}^{\infty} d\lambda_- \det_{m+1} S(\{\lambda_2, \dots, \lambda_m, \lambda_+, \lambda_-\}, \{\xi_j\}) \end{aligned}$$

$$\begin{aligned}
& \times \left(\prod_{j>k \geq 2}^m \frac{\sinh(\lambda_j - \xi_k + i\epsilon_j \zeta) \sinh(\lambda_k - \xi_j - i\epsilon_k \zeta)}{\sinh(\lambda_j - \lambda_k + i(\epsilon_j + \epsilon_k)\zeta)} \right) \\
& \times \frac{\sinh(\lambda_+ - \xi_1 + i\frac{\zeta}{2}) \sinh(\lambda_- - \xi_1 - i\frac{\zeta}{2})}{\sinh(\lambda_+ - \lambda_-)} \\
& \times \prod_{k=2}^m \frac{\sinh(\lambda_- - \xi_k - i\frac{\zeta}{2}) \sinh(\lambda_k - \xi_1 + i\epsilon_k \zeta)}{\sinh(\lambda_- - \lambda_k + i(\epsilon_k - \frac{1}{2})\zeta)} \\
& \times \prod_{k=2}^m \frac{\sinh(\lambda_+ - \xi_k + i\frac{\zeta}{2}) \sinh(\lambda_k - \xi_{m+1} - i\epsilon_k \zeta)}{\sinh(\lambda_+ - \lambda_k + i(\epsilon_k + \frac{1}{2})\zeta)}. \tag{4.2}
\end{aligned}$$

The blocks with the same number of operators D ($\epsilon_j = -\frac{1}{2}$) can be put together

$$g_{+-}(m) = \sum_{s=0}^{m-1} \tilde{g}_{+-}(m, s), \tag{4.3}$$

where

$$\begin{aligned}
\tilde{g}_{+-}(m, s) &= \left(\prod_{k=1}^{m+1} \prod_{j=1}^{M/2} \frac{\sinh(\lambda_j - \xi_k + i\frac{\zeta}{2})}{\sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})} \right) \sum_{a_2+\dots+a_m-m+1=s} \langle \psi_g | C \\
& \times \left(\xi_1 - i\frac{\zeta}{2} \right) T_{a_2 a_2} \left(\xi_1 - i\frac{\zeta}{2} \right) \cdots T_{a_m a_m} \left(\xi_m - i\frac{\zeta}{2} \right) B \left(\xi_{m+1} - i\frac{\zeta}{2} \right) | \psi_g \rangle. \tag{4.4}
\end{aligned}$$

It is easy to see that in all the terms of this sum written as multiple integrals the determinant of densities and *all the factors* containing λ_+ and λ_- are the same. Thus, to obtain the corresponding algebraic part it is sufficient to take the sum over all possible permutations of $\lambda_2, \dots, \lambda_m$ of the product

$$\prod_{j>k \geq 2}^m \frac{\sinh(\lambda_j - \xi_k + i\epsilon_j \zeta) \sinh(\lambda_k - \xi_j - i\epsilon_k \zeta)}{\sinh(\lambda_j - \lambda_k + i(\epsilon_j + \epsilon_k)\zeta)}.$$

This sum has exactly the same form as the corresponding term for the generating function. Using exactly the same arguments as in the previous section, we obtain for the contributions to the two-point function:

$$\begin{aligned}
\tilde{g}_{+-}(m, s) &= \frac{1}{s!(m-1-s)! \sin^{m-1} \zeta} \\
& \times \prod_{m+1 \geq k > j \geq 1} \frac{1}{\sinh(\xi_j - \xi_k)} \int_{-\infty}^{\infty} d\lambda_2 \cdots \int_{-\infty}^{\infty} d\lambda_m \int_{-\infty}^{\infty} d\lambda_+ \int_{-\infty}^{\infty} d\lambda_- \\
& \times \left(\prod_{k=2}^{s+1} \frac{\sinh(\lambda_- - \xi_k - i\frac{\zeta}{2}) \sinh(\lambda_k - \xi_1 - i\frac{\zeta}{2})}{\sinh(\lambda_- - \lambda_k - i\zeta)} \right) \\
& \times \left(\prod_{k=s+2}^m \frac{\sinh(\lambda_- - \xi_k - i\frac{\zeta}{2}) \sinh(\lambda_k - \xi_1 + i\frac{\zeta}{2})}{\sinh(\lambda_- - \lambda_k)} \right) \\
& \times \left(\prod_{k=2}^{s+1} \frac{\sinh(\lambda_+ - \xi_k + i\frac{\zeta}{2}) \sinh(\lambda_k - \xi_{m+1} + i\frac{\zeta}{2})}{\sinh(\lambda_+ - \lambda_k)} \right) \\
& \times \left(\prod_{k=s+2}^m \frac{\sinh(\lambda_+ - \xi_k + i\frac{\zeta}{2}) \sinh(\lambda_k - \xi_{m+1} - i\frac{\zeta}{2})}{\sinh(\lambda_+ - \lambda_k + i\zeta)} \right)
\end{aligned}$$

$$\begin{aligned} & \times \frac{\sinh(\lambda_+ - \xi_1 + i\frac{\zeta}{2}) \sinh(\lambda_- - \xi_1 - i\frac{\zeta}{2})}{\sinh(\lambda_+ - \lambda_-)} \cdot \Theta_{m-1}^s(\lambda_2, \dots, \lambda_m) \\ & \times Z_{m-1}(\{\lambda_2, \dots, \lambda_m\}|\{\xi_2, \dots, \xi_m\}) \\ & \times \det_{m+1} S(\{\lambda_2, \dots, \lambda_m, \lambda_-, \lambda_+\}, \{\xi_1, \dots, \xi_{m+1}\}). \end{aligned} \tag{4.5}$$

This result is slightly more complicated but very similar to the result for the generating function.

One should note that a very similar formula can be written for the correlation function $g_{zz}(m)$ directly (without any use of the generating function). This function can be written in the following form in terms of the monodromy matrix elements:

$$\begin{aligned} g_{zz}(m) = & -1 + 4 \left(\prod_{k=1}^{m+1} \prod_{j=1}^{M/2} \frac{\sinh(\lambda_j - \xi_k + i\frac{\zeta}{2})}{\sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})} \right) \\ & \times \langle \psi_g | D \left(\xi_1 - i\frac{\zeta}{2} \right) \left(\prod_{j=2}^m \left(A \left(\xi_j - i\frac{\zeta}{2} \right) + D \left(\xi_j - i\frac{\zeta}{2} \right) \right) \right) D \left(\xi_{m+1} - i\frac{\zeta}{2} \right) | \psi_g \rangle. \end{aligned} \tag{4.6}$$

To calculate this function one should sum up 2^{m-1} elementary blocks of the following form:

$$\begin{aligned} g_{zz}(m) = & -1 + 4 \left(\prod_{k=1}^{m+1} \prod_{j=1}^{M/2} \frac{\sinh(\lambda_j - \xi_k + i\frac{\zeta}{2})}{\sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})} \right) \\ & \times \sum_{a_j=1,2} \langle \psi_g | D \left(\xi_1 - i\frac{\zeta}{2} \right) \left(\prod_{j=2}^m T_{a_j a_j} \left(\xi_j - i\frac{\zeta}{2} \right) \right) D \left(\xi_{m+1} - i\frac{\zeta}{2} \right) | \psi_g \rangle. \end{aligned}$$

As usual we put together the blocks with the same number of operators D ,

$$g_{zz}(m) = -1 + 4 \sum_{s=0}^{m-1} \tilde{g}_{DD}(m, s), \tag{4.7}$$

where

$$\begin{aligned} \tilde{g}_{DD}(m, s) = & \left(\prod_{k=1}^{m+1} \prod_{j=1}^{M/2} \frac{\sinh(\lambda_j - \xi_k + i\frac{\zeta}{2})}{\sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})} \right) \sum_{a_2+\dots+a_m-m+1=s} \\ & \times \langle \psi_g | D \left(\xi_1 - i\frac{\zeta}{2} \right) T_{a_2 a_2} \left(\xi_1 - i\frac{\zeta}{2} \right) \dots T_{a_m a_m} \left(\xi_m - i\frac{\zeta}{2} \right) D \left(\xi_{m+1} - i\frac{\zeta}{2} \right) | \psi_g \rangle. \end{aligned} \tag{4.8}$$

Using exactly the same arguments as for the function $g_{+-}(m)$ we obtain for the contributions to the two-point function:

$$\begin{aligned} \tilde{g}_{DD}(m, s) = & \frac{1}{s!(m-1-s)! \sin^{m-1} \zeta} \\ & \times \prod_{m+1 \geq k > j \geq 1} \frac{1}{\sinh(\xi_j - \xi_k)} \int_{-\infty}^{\infty} d\lambda_1 \int_{-\infty}^{\infty} d\lambda_2 \dots \int_{-\infty}^{\infty} d\lambda_m \int_{-\infty}^{\infty} d\lambda_{m+1} \\ & \times \left(\prod_{k=2}^{s+1} \frac{\sinh(\lambda_1 - \xi_k + i\frac{\zeta}{2}) \sinh(\lambda_k - \xi_1 - i\frac{\zeta}{2})}{\sinh(\lambda_k - \lambda_1 - i\zeta)} \right) \end{aligned}$$

$$\begin{aligned}
& \times \left(\prod_{k=s+2}^m \frac{\sinh(\lambda_1 - \xi_k + i\frac{\zeta}{2}) \sinh(\lambda_k - \xi_1 + i\frac{\zeta}{2})}{\sinh(\lambda_k - \lambda_1)} \right) \\
& \times \left(\prod_{k=2}^{s+1} \frac{\sinh(\lambda_{m+1} - \xi_k - i\frac{\zeta}{2}) \sinh(\lambda_k - \xi_{m+1} + i\frac{\zeta}{2})}{\sinh(\lambda_{m+1} - \lambda_k - i\zeta)} \right) \\
& \times \left(\prod_{k=s+2}^m \frac{\sinh(\lambda_{m+1} - \xi_k - i\frac{\zeta}{2}) \sinh(\lambda_k - \xi_{m+1} - i\frac{\zeta}{2})}{\sinh(\lambda_{m+1} - \lambda_k)} \right) \\
& \times \frac{\sinh(\lambda_{m+1} - \xi_1 - i\frac{\zeta}{2}) \sinh(\lambda_1 - \xi_{m+1} + i\frac{\zeta}{2})}{\sinh(\lambda_{m+1} - \lambda_1 - i\zeta)} \cdot \Theta_{m-1}^s(\lambda_2, \dots, \lambda_m) \\
& \times Z_{m-1}(\{\lambda_2, \dots, \lambda_m\}|\{\xi_2, \dots, \xi_m\}) \\
& \times \det_{m+1} S(\{\lambda_1, \lambda_2, \dots, \lambda_m, \lambda_{m+1}\}, \{\xi_1, \dots, \xi_{m+1}\}). \tag{4.9}
\end{aligned}$$

5. Further re-summations

In this section, we show how one can proceed to a complete re-summation of terms and reduce the result for the two-point correlation functions to only one term (written again as a multiple integral). To do it one should note that all the terms in the sum (3.5) have a very similar structure. Extracting the common denominator, one can obtain the following sum for the generating function:

$$\begin{aligned}
\mathcal{Q}_m(\beta) &= \sum_{s=0}^m \frac{e^{s\beta}}{s!(m-s)! \sin^m \zeta} \prod_{j < k} \frac{1}{\sinh(\xi_j - \xi_k)} \int_{-\infty}^{\infty} d\lambda_1 \cdots \int_{-\infty}^{\infty} d\lambda_m \\
& \times \frac{\prod_{k=1}^m \prod_{j=1}^m \sinh(\lambda_j - \xi_k + i\frac{\zeta}{2}) \sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})}{\prod_{j > k} \sinh(\lambda_j - \lambda_k) \sinh(\lambda_j - \lambda_k + i\zeta) \sinh(\lambda_j - \lambda_k - i\zeta)} \\
& \times Z_m(\{\lambda\}|\{\xi\}) \det_m S(\{\lambda\}, \{\xi\}) H_s(\{\lambda\}|\{\xi\}) \bar{H}_s(\{\lambda\}|\{\xi\}), \tag{5.1}
\end{aligned}$$

where \bar{H}_s means complex conjugation and the function $H_s(\{\lambda\}|\{\xi\})$ is defined as

$$\begin{aligned}
& H_s(\{\lambda\}|\{\xi\}) \\
&= \frac{\prod_{j=1}^s \prod_{k=s+1}^m \sinh(\lambda_j - \lambda_k + i\zeta) \prod_{1 \leq j < k \leq s} \sinh(\lambda_j - \lambda_k) \prod_{s+1 \leq j < k \leq m} \sinh(\lambda_j - \lambda_k)}{\prod_{k=1}^m (\prod_{j=1}^s \sinh(\lambda_j - \xi_k - i\frac{\zeta}{2}) \prod_{j=s+1}^m \sinh(\lambda_j - \xi_k + i\frac{\zeta}{2}))}. \tag{5.2}
\end{aligned}$$

It is easy to see that this function can be written as a Cauchy determinant

$$\begin{aligned}
H_s(\{\lambda\}|\{\xi\}) &= \prod_{j < k} \frac{1}{\sinh(\xi_k - \xi_j)} \det \mathcal{H}^{(s)}, \\
\mathcal{H}_{jk}^{(s)} &= \frac{1}{\sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})}, \quad j \leq s, \\
\mathcal{H}_{jk}^{(s)} &= \frac{1}{\sinh(\lambda_j - \xi_k + i\frac{\zeta}{2})}, \quad j > s.
\end{aligned}$$

The sum over s in (5.1) can be taken under the integrals. Here, we separated the terms which depend on s :

$$\sum_{s=0}^m \frac{e^{\beta s}}{s!(m-s)!} \det \mathcal{H}^{(s)} \det \bar{\mathcal{H}}^{(s)}.$$

This sum can be simplified if one introduce some auxiliary contour integrals:

$$\sum_{s=0}^m \frac{e^{\beta s}}{s!(m-s)!} \det \mathcal{H}^{(s)} \det \bar{\mathcal{H}}^{(s)} = \frac{1}{m!} \oint \frac{dz_1}{2i\pi} \cdots \oint \frac{dz_m}{2i\pi} \times \exp \left(\frac{\beta}{i\zeta} \sum_{j=1}^m \left(z_j + i\frac{\zeta}{2} \right) \right) \left(\prod_{j=1}^m \frac{\sinh 2z_j}{\sinh(z_j - i\frac{\zeta}{2}) \sinh(z_j + i\frac{\zeta}{2})} \right) \det \mathcal{F}^+ \det \mathcal{F}^-,$$

where

$$\mathcal{F}_{jk}^{\pm} = \frac{1}{\sinh(\lambda_j \pm z_j - \xi_k)},$$

and contours are chosen in such a way that the points $z_j = \pm i\frac{\zeta}{2}$ are inside the contours and all the poles at the points $\lambda_j \pm z_j - \xi_k = 0$ are outside. Thus, we can represent the generating function as a single term but the number of integrals is now $2m$:

$$\begin{aligned} \mathcal{Q}_m(\beta) &= \frac{1}{m! \sin^m \zeta} \prod_{j < k} \frac{1}{\sinh^3(\xi_j - \xi_k)} \int_{-\infty}^{\infty} d\lambda_1 \cdots \int_{-\infty}^{\infty} d\lambda_m \oint \frac{dz_1}{2i\pi} \cdots \oint \frac{dz_m}{2i\pi} Z_m(\{\lambda\}|\{\xi\}) \\ &\times \frac{\prod_{k=1}^m \prod_{j=1}^m \sinh(\lambda_j - \xi_k + i\frac{\zeta}{2}) \sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})}{\prod_{j > k} \sinh(\lambda_j - \lambda_k) \sinh(\lambda_j - \lambda_k + i\zeta) \sinh(\lambda_j - \lambda_k - i\zeta)} \det_m S(\{\lambda\}, \{\xi\}) \\ &\times \left(\prod_{j=1}^m \frac{\exp\left(\frac{\beta}{i\zeta}\left(z_j + i\frac{\zeta}{2}\right)\right) \sinh 2z_j}{\sinh(z_j - i\frac{\zeta}{2}) \sinh(z_j + i\frac{\zeta}{2})} \right) \det \mathcal{F}^+ \det \mathcal{F}^-. \end{aligned} \tag{5.3}$$

Now, using the symmetry of the expression under the integral with respect to permutations of pairs (λ_j, z_j) , we can replace one of the determinants $\det \mathcal{F}^{\pm}$ by a product of its diagonal terms. It permits us in particular to separate the variables z_j and hence to integrate over them. It is easy to see that it leads to the following final result for the generating function:

$$\begin{aligned} \mathcal{Q}_m(\beta) &= \frac{1}{\sin^m \zeta} \prod_{j < k} \frac{1}{\sinh^3(\xi_j - \xi_k)} \int_{-\infty}^{\infty} d\lambda_1 \cdots \int_{-\infty}^{\infty} d\lambda_m Z_m(\{\lambda\}|\{\xi\}) \det_m S(\{\lambda\}, \{\xi\}) \\ &\times \frac{\prod_{k=1}^m \prod_{j=1}^m \sinh(\lambda_j - \xi_k + i\frac{\zeta}{2}) \sinh(\lambda_j - \xi_k - i\frac{\zeta}{2})}{\prod_{j > k} \sinh(\lambda_j - \lambda_k) \sinh(\lambda_j - \lambda_k + i\zeta) \sinh(\lambda_j - \lambda_k - i\zeta)} \det_m \mathcal{G}, \end{aligned} \tag{5.4}$$

where the $m \times m$ matrix \mathcal{G} is defined as

$$\mathcal{G}_{jk} = \frac{e^{\beta}}{\sinh(\lambda_j - \xi_k + i\frac{\zeta}{2}) \sinh(\lambda_j - \xi_j - i\frac{\zeta}{2})} + \frac{1}{\sinh(\lambda_j - \xi_k - i\frac{\zeta}{2}) \sinh(\lambda_j - \xi_j + i\frac{\zeta}{2})}. \tag{5.5}$$

Similar results can be obtained for the two-point functions. This is the most compact formula for the generating function (we reduced the number of terms from 2^m to one). However, we think that the formulae obtained in the two previous sections are more convenient for the asymptotic analysis as they permit a natural homogeneous limit (which is not the case of (5.4)).

6. Free-fermion point

As the first application and check of this new re-summation formula, we consider the free-fermion point ($\zeta = \frac{\pi}{2}$). Of course the representations for the correlation functions in this point have already been obtained by different methods, but the formulae (3.13) and (4.5) give a simple and elegant way to get these explicit results.

6.1. Generating function

We calculate the generating function

$$\mathcal{Q}_m(\beta) \equiv \langle \psi_g | \exp\{\beta \mathcal{Q}_{1,m}\} | \psi_g \rangle, \quad \mathcal{Q}_{1,m} = \sum_{j=1}^m \frac{1}{2} (1 - \sigma_j^z).$$

This function can be written as

$$\mathcal{Q}_m(\beta) = \sum_{s=0}^m e^{s\beta} F_s(m), \quad (6.1)$$

where contributions $F_s(m)$ are given by (3.13).

Taking into account that $\zeta = \frac{\pi}{2}$ all the determinants can be calculated. In the homogeneous limit, one obtains the following multiple integral representation:

$$\begin{aligned} F_s(m) = & \frac{2^{m^2-m}}{\pi^m s! (m-s)!} \int_{-\infty}^{\infty} d\lambda_1 \cdots \int_{-\infty}^{\infty} d\lambda_m \prod_{1 < j < k \leq s} \sinh^2(\lambda_j - \lambda_k) \\ & \times \prod_{s < j < k \leq m} \sinh^2(\lambda_j - \lambda_k) \prod_{j=1}^s \prod_{k=s+1}^m \cosh^2(\lambda_j - \lambda_k) \prod_{j=1}^m \cosh^{-m} 2\lambda_j. \end{aligned} \quad (6.2)$$

After changing variables

$$\lambda_j = \frac{1}{2} \log(\tan p_j),$$

we obtain a much simpler representation:

$$F_s(m) = \frac{2^{m^2-m}}{\pi^m s! (m-s)!} \int_0^{\frac{\pi}{2}} dp_1 \cdots \int_0^{\frac{\pi}{2}} dp_m \prod_{j>k} \sin^2(\varepsilon_j p_j - \varepsilon_k p_k), \quad (6.3)$$

where $\varepsilon_j = 1$ for $j \leq s$ and $\varepsilon_j = -1$ for $j > s$. The expression under the integral can be rewritten as a product of two Vandermonde determinants:

$$\begin{aligned} \prod_{j>k} \sin^2(\varepsilon_j p_j - \varepsilon_k p_k) &= 2^{m-m^2} \left| \prod_{j>k} (e^{2i\varepsilon_j p_j} - e^{2i\varepsilon_k p_k}) \right|^2 \\ &= 2^{m-m^2} \det V(\{\varepsilon p\}) \det V^*(\{\varepsilon p\}), \\ V_{jk}(\{p\}) &= e^{2i(k-1)p_j}, \end{aligned}$$

where the star means Hermitian conjugation. The product of these two determinants can be calculated as the determinant of the product of two matrices:

$$\begin{aligned} \det V(\{\varepsilon p\}) \det V^*(\{\varepsilon p\}) &= \det (V^*(\{\varepsilon p\}) V(\{\varepsilon p\})) \\ (V^* V)_{kn} &= \sum_{j=1}^m e^{2i(k-n)\varepsilon_j p_j}. \end{aligned}$$

Now we should take into account that we integrate this determinant. It can be written as a sum over permutations,

$$F_s(m) = \frac{1}{\pi^m s! (m-s)!} \int_0^{\frac{\pi}{2}} dp_1 \cdots \int_0^{\frac{\pi}{2}} dp_m \sum_{\sigma} \det W(\sigma, s, \{p\}),$$

where the matrix W is defined as

$$W_{kn}(\sigma, s, \{p\}) = e^{2i(k-n)\varepsilon_{\sigma(n)} p_{\sigma(n)}}.$$

As we integrate over all the variables p , permutations of the variables with the same value of the parameter ε do not change the integration result and thus the sum over permutations can be replaced by a sum over the partitions of the set $\{p\}$ into two subsets $\{p^+\}$ and $\{p^-\}$, with the number of elements in the first one being s ,

$$F_s(m) = \frac{1}{\pi^m} \int_0^{\frac{\pi}{2}} dp_1 \cdots \int_0^{\frac{\pi}{2}} dp_m \sum_{\{p\}=\{p^+\} \cup \{p^-\}} \det \tilde{W}(\{p^+\}, \{p^-\}).$$

Here, the matrix $\tilde{W}(\{p^+\}, \{p^-\})$ is defined as

$$\tilde{W}_{kn}(\{p^+\}, \{p^-\}) = e^{2i(k-n)\varepsilon_n p_n^{\varepsilon_n}},$$

ε_n being + or –.

Now considering the entire sum over s (6.1), one can easily note that it can be rewritten as a determinant of a sum of two matrices:

$$\begin{aligned} Q_m(\beta) &= \frac{1}{\pi^m} \int_0^{\frac{\pi}{2}} dp_1 \cdots \int_0^{\frac{\pi}{2}} dp_m \det_m U(\beta, \{p\}), \\ U_{kn}(\beta, \{p\}) &= e^{2i(k-n)p_n} + e^\beta e^{2i(n-k)p_n}. \end{aligned} \tag{6.4}$$

It is now possible to calculate all the integrals and to write the final result as a determinant:

$$\begin{aligned} Q_m(\beta) &= \det_m T(\beta), \\ T_{kn}(\beta) &= \delta_{kn} \frac{e^\beta + 1}{2} + (1 - \delta_{kn})(1 - e^\beta) \frac{1 - (-1)^{n-k}}{2i\pi(n-k)}. \end{aligned} \tag{6.5}$$

From this formula, one can easily obtain the well-known result for the two-point function $g_{zz}(m)$:

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = \frac{2}{\pi^2 m^2} ((-1)^m - 1). \tag{6.6}$$

The emptiness formation probability [33] comes directly from (6.5) by taking the limit $\beta \rightarrow -\infty$.

6.2. Two-point functions

The two-point functions $\langle \sigma_1^+ \sigma_{m+1}^- \rangle$ and $\langle \sigma_1^- \sigma_{m+1}^+ \rangle$ can be calculated in a similar way in the free-fermion point.

First of all a re-summation formula can be obtained for these functions almost in the same way as for the generating function. We can consider even a more general function:

$$g_{+-}(m, \beta) = \langle \sigma_1^+ \exp(\beta Q_{2,m}) \sigma_{m+1}^- \rangle,$$

where $Q_{2,m} = \sum_{j=2}^m \frac{1}{2}(1 - \sigma_j^z)$, which also includes the correlation function of fermionic fields. After re-summation it can be represented as

$$g_{+-}(m, \beta) = \sum_{s=0}^{m-1} e^{s\beta} g_s^{+-}(m), \tag{6.7}$$

where the contributions $g_s^{+-}(m)$ are given by (4.5). This general re-summation formula can be simplified in the free-fermion case:

$$\begin{aligned} g_s^{+-}(m) &= \frac{2^{m^2+1} (-1)^{s+1}}{i^{m-1} \pi^{m+1} s! (m-1-s)!} \int_{-\infty}^{\infty} d\lambda_2 \cdots \int_{-\infty}^{\infty} d\lambda_m \int_{-\infty}^{\infty} d\lambda_+ \int_{-\infty}^{\infty} d\lambda_- \\ &\times \frac{\prod_{2 < j < k \leq s+1} \sinh^2(\lambda_j - \lambda_k) \prod_{s+1 < j < k \leq m} \sinh^2(\lambda_j - \lambda_k) \prod_{j=1}^{s+1} \prod_{k=s+2}^m \cosh^2(\lambda_j - \lambda_k)}{\prod_{j=2}^m \cosh^m 2\lambda_j} \end{aligned}$$

$$\begin{aligned} & \times \frac{\cosh(\lambda_+ - \lambda_-) \prod_{j=2}^{s+1} \sinh(\lambda_+ - \lambda_j) \cosh(\lambda_- - \lambda_j) \prod_{j=s+2}^m \cosh(\lambda_+ - \lambda_j) \sinh(\lambda_- - \lambda_j)}{\cosh^{m+1} 2\lambda_+ \cosh^{m+1} 2\lambda_-} \\ & \times \sinh^m \left(\lambda_+ - i\frac{\pi}{4} \right) \sinh^m \left(\lambda_- + i\frac{\pi}{4} \right). \end{aligned} \tag{6.8}$$

Changing variables as in the previous case

$$\lambda_j = \frac{1}{2} \log(\tan p_j),$$

we obtain

$$\begin{aligned} g_s^{+-}(m) &= \frac{2^{m^2-m+1} (-1)^{s+1}}{i^{m-1} \pi^{m+1} s! (m-1-s)!} \int_0^{\frac{\pi}{2}} dp_2 \cdots \int_0^{\frac{\pi}{2}} dp_m \int_0^{\frac{\pi}{2}} dp_+ \int_0^{\frac{\pi}{2}} dp_- e^{i(p_+ - p_-)m} \\ & \times \prod_{m \geq j > k > 1} \sin^2(\varepsilon_j p_j - \varepsilon_k p_k) \prod_{j=2}^m (\sin(p_+ - \varepsilon_j p_j) \sin(p_- + \varepsilon_j p_j)) \sin(p_+ + p_-), \end{aligned} \tag{6.9}$$

where $\varepsilon_j = 1$ for $j \leq s + 1$ and $\varepsilon_j = -1$ for $j > s + 1$. This expression can also be rewritten as a product of two Vandermonde determinants, but now the matrices have different sizes:

$$\begin{aligned} g_s^{+-}(m) &= \frac{(-1)^s}{i^m \pi^{m+1} s! (m-1-s)!} \int_0^{\frac{\pi}{2}} dp_2 \cdots \int_0^{\frac{\pi}{2}} dp_m \int_0^{\frac{\pi}{2}} dp_+ \int_0^{\frac{\pi}{2}} dp_- e^{-2i \sum_{j=2}^m \varepsilon_j p_j} \\ & \times \det_{m+1} V(m+1, \{p_+, -p_-, \varepsilon_2 p_2, \dots, \varepsilon_m p_m\}) \\ & \times \det_{m-1} V^*(m-1, \{\varepsilon_2 p_2, \dots, \varepsilon_m p_m\}), \\ V_{jk} &= e^{2i(k-1)\varepsilon_j p_j}. \end{aligned}$$

The product of these two determinants can be again rewritten as a determinant of a product of two matrices if we add two rows and two columns to the second one:

$$\begin{aligned} \tilde{V}_{1j}^*(m+1, \{p\}) &= \tilde{V}_{j1}^*(m+1, \{p\}) = \delta_{1j}, \\ \tilde{V}_{2j}^*(m+1, \{p\}) &= \tilde{V}_{j2}^*(m+1, \{p\}) = \delta_{2j}, \\ \tilde{V}_{kj}^*(m+1, \{p\}) &= V_{k-2j-2}^*(m-1, \{p\}), \quad k, j > 2. \end{aligned}$$

Now we can take the product of these two determinants and proceed with the sum over s in the same way as for the generating function:

$$\begin{aligned} g_{+-}(m, \beta) &= \frac{1}{i^m \pi^{m+1}} \int_0^{\frac{\pi}{2}} dp_2 \cdots \int_0^{\frac{\pi}{2}} dp_m \int_0^{\frac{\pi}{2}} dp_+ \int_0^{\frac{\pi}{2}} dp_- \det_{m+1} \tilde{U}(\beta, \{p\}), \\ \tilde{U}_{1n}(\beta, \{p\}) &= e^{2i(n-1)p_+}, \\ \tilde{U}_{2n}(\beta, \{p\}) &= e^{-2i(n-1)p_-}, \\ U_{kn}(\beta, \{p\}) &= e^{2i(k-n-1)p_n} - e^\beta e^{2i(n-k+1)p_n}. \end{aligned} \tag{6.10}$$

Now all the integrals can be calculated. For the two-point function ($\beta = 0$), we get

$$\begin{aligned} g_{+-}(m) &= \frac{(-1)^m}{\pi^{m+1}} \det_{m+1} T^{-+}(m), & T_{11}^{-+} &= T_{21}^{-+} = \frac{\pi}{2}, \\ T_{1n}^{-+} &= -T_{2n}^{-+} = \frac{1 + (-1)^n}{2(n-1)}, & T_{kn}^{-+} &= \frac{1 + (-1)^{k-n}}{(k-n-1)}. \end{aligned} \tag{6.11}$$

This determinant can be easily computed and gives a formula obtained by Wu in [7] (a more general formula was obtained by McCoy [8]):

$$\langle \sigma_1^+ \sigma_{m+1}^- \rangle = \frac{(-1)^m}{2} \prod_{k=1}^{\lfloor \frac{m}{2} \rfloor} \frac{\Gamma^2(k)}{\Gamma(k - \frac{1}{2})\Gamma(k + \frac{1}{2})} \prod_{k=1}^{\lfloor \frac{m+1}{2} \rfloor} \frac{\Gamma^2(k)}{\Gamma(k - \frac{1}{2})\Gamma(k + \frac{1}{2})}. \quad (6.12)$$

The asymptotic behaviour of such a product was also obtained in [7] using the technique introduced in [34]:

$$\langle \sigma_1^+ \sigma_{m+1}^- \rangle = \frac{(-1)^m}{\sqrt{2m}} \exp \left\{ \frac{1}{2} \int_0^\infty \frac{dt}{t} \left[e^{-4t} - \frac{1}{\cosh^2 t} \right] \right\} \left(1 - \frac{(-1)^m}{8m^2} + \mathcal{O}(m^{-4}) \right). \quad (6.13)$$

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